2.2 - Separable Equations

Definition: A **differential equation** (DE) is an equation containing the derivatives of one or more unknown functions (dependent variables) with respect to one or more independent variables. $y(x)$

x

Examples: $\dfrac{dy}{dx} + 5y = e^x$, $y'' - y' + 6y = 0$, and $\dfrac{dx}{dt} + \dfrac{dy}{dt} = 2x + y$ are ordinary differential equations (ODEs); $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ is an example of a partial differential equation (PDE). In this course, we will only work with ODEs.

Definition: The **order** of a DE is the order of the highest derivative in the equation.

$y' = \frac{dy}{dx} = f(x,y)$

Definition: A first-order **separable** DE has the form $\dfrac{dy}{dx} = g(x)h(y)$.

To solve, we separate variables and integrate:

$$\frac{1}{h(y)}\,dy = g(x)\,dx \quad \text{OR} \quad P(y)\,dy = g(x)\,dx$$

$$\int P(y)\,dy = \int g(x)\,dx$$

Example: Solve the given differential equation by separation of variables.

$dy - (y-1)^2 dx = 0$

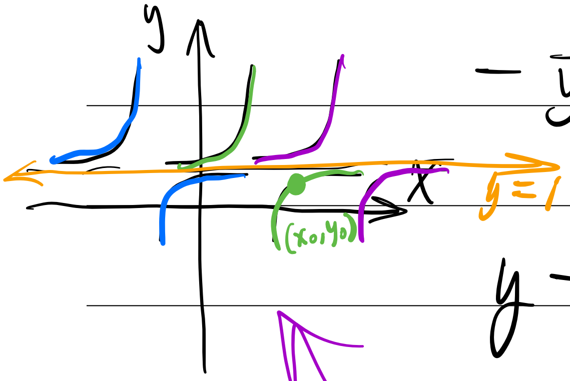
Note: This is a nonlinear DE because the equation contains a nonlinear function of y (the dependent variable).

$$dy = (y-1)^2\,dx$$

$$(y-1)^{-2}\,dy = dx$$

$$\int (y-1)^{-2}\,dy = \int dx$$

$y = 1$ is a <u>singular solution</u>.



$$-\frac{1}{y-1} = x + C$$

Note: we only need one constant of integration

$$y - 1 = -\frac{1}{x+C}$$

$$y = 1 - \frac{1}{x+C}$$

arbitrary constant

one-parameter family of solutions

Since y is isolated, this is an explicit solution.

Definition: Any function ϕ (Phi), defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ODE reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

Note: The function must be diff'able and so cont. on an interval of solution

Example: $\sin 3x \; dx + 2y \cos^3 3x \; dy = 0$

since one term has $y\,dy$, this is nonlinear

$$\int 2y\,dy = -\frac{1}{3}\int \frac{3\sin 3x}{\cos^3 3x}\,dx$$

$u = \cos 3x$

$du = -3\sin 3x\,dx$

$$y^2 = \frac{1}{3}\left(-\frac{1}{2}\right)\cos^{-2} 3x + C$$

$$\boxed{y^2 = -\frac{1}{6}\sec^2 3x + C}$$

This is an implicit solution

Example: $\frac{dN}{dt} + N = Ne^{t+2}$ $\Longrightarrow$ $\frac{dN}{dt} = Ne^{t+2} - N$

Note: This is a linear DE because all instances of N (the dependent variable) are linear.

$$\int \frac{dN}{N} = \int (e^{t+2} - 1)\,dt$$

$$\ln|N| = \underline{e^{t+2} - t} + \underline{C_1}$$

$$e^{x+y}$$

$$e^x e^y$$

$$|N| = \underline{e^{e^{t+2} - t}}\ \underline{e^{C_1}}$$

$$N = c\, e^{e^{t+2} - t}$$

Let $c = \pm e^{C_1}$

Definition: Information such as $y(x_0) = y_0$ is called an **initial condition**. A DE, together with an initial condition is called an **initial-value problem** (IVP).

Example: Find an explicit solution of the given initial-value problem. Determine the exact interval I of definition by analytical methods. Use a graphing utility to plot the graph of the function.

$$(2y - 2)\frac{dy}{dx} = 3x^2 + 4x + 2, \qquad y(1) = -2$$

When $x = 1$, $y = -2$

$$(2y-2)\,dy = (3x^2+4x+2)\,dx$$

$C = C_1 + 1$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C_1$$

$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + C$$

$$9 = 5 + C \Rightarrow C = 4$$

$$(y-1)^2 = x^3+2x^2+2x+4$$

$$y-1 = \pm\sqrt{x^3+2x^2+2x+4}$$

$$y' = -\frac{3x^2+2x+2}{\sqrt{x^3+2x^2+2x+4}}$$

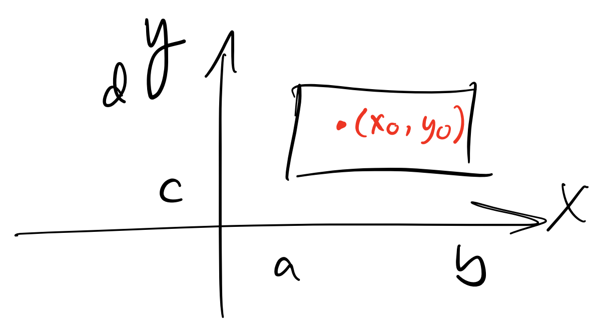
$$y = 1 - \sqrt{x^3+2x^2+2x+4}$$

Interval: $x^3+2x^2+2x+4 > 0$

$$x^2(x+2)+2(x+2) > 0$$

$$(x^2+2)(x+2) > 0 \Rightarrow x > -2$$

Interval: $(-2, \infty)$



Theorem: Existence of a Unique Solution

Consider the first-order initial-value problem

Solve $\dfrac{dy}{dx} = f(x, y)$, subject to $y(x_0) = y_0$

Let R be a rectangular region in the xy-plane defined by $a \le x \le b,\ c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on R, then there exists some interval $I_0 : (x_0 - h, x_0 + h),\ h > 0$, contained in $[a,\ b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the initial-value problem.

We will consider higher-order DEs later in the course.

Example: Find an explicit solution of the given initial-value problem. Give the solution using a definite integral.

$$\frac{dy}{dx} = y^2 \sin x^2, \qquad y(-2) = \frac{1}{3}$$

$$\int y^{-2}\,dy = \int \sin x^2\,dx$$

$$\int_{-2}^{x} y^{-2}\,dy = \int_{-2}^{x} \sin t^2\,dt$$

t is a dummy variable

$$-\frac{1}{y(t)}\Bigg|_{-2}^{x} = \int_{-2}^{x} \sin t^2\,dt$$

$$-\frac{1}{y(x)} + \frac{1}{y(-2)} = \int_{-2}^{x} \sin t^2\,dt$$

$y(-2) = \frac{1}{3}$

$$-\frac{1}{y(x)} + 3 = \int_{-2}^{x} \sin t^2\,dt$$

$$y(x) = \left(3 - \int_{-2}^{x} \sin t^2\,dt\right)^{-1}$$

Calc – start w/ function $\xrightarrow{\text{limits}}$ rate

accumulated values

DE – start with info about a rate → function that fits

↪ model